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In this thesis the author discusses the topological equivalence of spaces imbedded in the plane. The material is divided into four categories: compactness, connectedness, points of separation, and components. Examples of homeomorphic and non-homeomorphic spaces are presented in in each category.

HOMEOMORPHIC SPACES IN THE PLANE

by

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INTRODUCTION

The idea of topological equivalence, or homeomorphism, is one of the most basic in any study of topology. In this paper, pairs of spaces imbedded in the plane are compared to determine whether or not they are homeomorphic. The decisions are based on four main properties.

In Chapter I, homeomorphism is defined and the conditions under which a property is said to be a topological property are given.

Chapter II deals with compactness, the first of the four properties previously mentioned. Examples of compact and non-compact spaces are given and then used to investigate the topological equivalence of some well-known spaces.

The second of the four properties, connectedness, is defined and discussed in Chapter III. It is shown that connectedness is a topological property, and examples of homeomorphic and non-homeomorphic spaces are given.

The third of the four properties, that of having a point of separation, is presented in Chapter IV. It is proved that this is indeed a topological property, and examples of sets with this property are studied.

In Chapter V, an investigation of components is made. It is proved that there must be a one-to-one correspondence between the components of two homeomorphic spaces, and this theorem is used in determining topological equivalence in several examples.

When not specifically stated, it will be understood that the set under consideration has the usual topology inherited from the reals or from the plane, whichever is applicable. Throughout the paper the symbol " $\backsim$ " will mean "is homeomorphic to." The reader is referred to [1] for definitions and theorems not covered in this paper.

CHAPTER I

ELEMENTS OF TOPOLOGICAL EQUIVALENCE

Definition 1: Let (X, S) and (Y, T) be topological spaces. We say (X, S) is homeomorphic to (Y, T), written (X, S) $\backsim$ (Y, T), provided there is a mapping

$$f: (X, S) \rightarrow (Y, T)$$

such that f is one-to-one, onto, and continuous, and f^{-1} is continuous.

Theorem 1: The relation $\backsim$ is an equivalence relation.

Proof: Let (X, S), (Y, T), and (Z, R) be topological spaces. We must show that

(1) (X, S) $\backsim$ (X, S),

(2) if (X, S) $\backsim$ (Y, T), then (Y, T) $\backsim$ (X, S), and

(3) if (X, S) $\backsim$ (Y, T) and (Y, T) $\backsim$ (Z, R), then (X, S) $\backsim$ (Z, R).

(1) Let f be the identity map, that is, for all $x \in X$, $f(x) = x$. Then f is certainly one-to-one, onto, and continuous; and since f^{-1} is also the identity map, f^{-1} is continuous. Hence, f is a homeomorphism and (X, S) $\backsim$ (X, S).

(2) Suppose (X, S) $\backsim$ (Y, T). Then there is a mapping f: (X, S) $\rightarrow$ (Y, T) such that f is one-to-one, onto, and continuous, and f^{-1} is continuous. Consider the

map f^{-1}: $(Y, T) \rightarrow (X, S)$. This mapping is one-to-one, onto, and continuous, and $(f^{-1})^{-1} = f$ is continuous. Therefore, $(Y, T) \backsim (X, S)$.

(3) Let f: $(X, S) \rightarrow (Y, T)$ and g: $(Y, T) \rightarrow (Z, R)$ be homeomorphisms. Then each of f and g is one-to-one, onto, and continuous, and both f^{-1} and g^{-1} are continuous. Consider the mapping $g \circ f$: $(X, S) \rightarrow (Z, R)$.

Suppose $(g \circ f)(x_1)$ and $(g \circ f)(x_2) \in Z$ such that

$$(g \circ f)(x_1) = (g \circ f)(x_2)$$

That is,

$$g(f(x_1)) = g(f(x_2))$$

Then

$$f(x_1) = f(x_2)$$

since g is one-to-one, and

$$x_1 = x_2$$

since f is one-to-one. Therefore, $(g \circ f)$ is one-to-one.

Now let $z \in Z$. Then there is an element $y \in Y$ such that $g(y) = z$, since g is onto. Since f is also onto, there is an element $x \in X$ such that $f(x) = y$. Then $g(f(x)) = g(y) = z$. Thus, $g \circ f$ is onto.

Since both f and g are continuous, $g \circ f$ is continuous. Also, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ is continuous because both f^{-1} and g^{-1} are.

Therefore $g \circ f$ is a homeomorphism, that is,

$(X, S) \backsim (Z, R)$.

Hence $\backsim$ is an equivalence relation.

Definition 2: A topological property C is a property such that if (X, S) and (Y, T) are topological spaces and $(X, S) \backsim (Y, T)$, then (X, S) has property C if and only if (Y, T) has property C.

CHAPTER II

COMPACTNESS

Definition 3: A topological space (X, T) is said to be compact provided every open cover of X has a finite subcover.

Definition 4: Let (X, T) be a topological space. A subset A of X is said to be compact if (A, T_A) is compact.

Theorem 2: Compactness is a topological property.

Proof: Let (X, S) and (Y, T) be topological spaces with $(X, S) \backsim (Y, T)$, and suppose (X, S) is compact.

Let C* be an open cover for Y. Since f is continuous, $f^{-1}(C)$ is open in X for each $C \in C^*$. Then $\left\{f^{-1}(C) \mid C \in C^*\right\}$ is an open cover for X. This cover has a finite subcover, say $f^{-1}(C_1)$, $f^{-1}(C_2)$, . . . $f^{-1}(C_n)$. Now f is one-to-one and onto, so $f(f^{-1}(C_i)) = C_i$, $(1 \leq i \leq n)$, and thus C_1, C_2, . . . C_n is a cover for Y.

Therefore Y is compact.

The proof that if (Y, T) is compact then (X, S) is also compact is entirely analogous. Hence compactness is a topological property.

Theorem 3: The closed unit interval [0, 1] is compact.

Proof: Let (X, T) denote the reals with the usual topology and consider $([0, 1], T_{[0, 1]})$. Let C* be any open cover for [0, 1].

Now let $U = \left\{x \mid x \in [0, 1] \text{ and } [0, x] \text{ can be covered by a finite number of elements of } C^*\right\}$. We know that U is not the empty set since $x = 0 \in U$. Let $v = \sup U$. Now suppose $v < 1$, so that [0, 1] cannot be covered by a finite number of elements of C*. Since $v \in [0, 1]$, there is a real number $n > 0$ such that $v + n = 1$. Also, since $v \in [0, 1]$, there is an element $V \in C^*$ such that $v \in V$. V is open, so there is a real number $\epsilon > 0$ such that $(v - \epsilon, v + \epsilon) \subset V$. Let $\lambda = \min(n, \epsilon)$. Then $[v - \frac{\lambda}{2}, v + \frac{\lambda}{2}] \subset (v - \epsilon, v + \epsilon) \subset V$.

Claim: $v \in U$.

Subproof: Since $v \in [0, 1]$, there is an element $C_v \in C^*$ such that $v \in C_v$. Now C_v is open, so there is a real number $k > 0$ such that $(v - k, v + k) \subset C_v$.

Suppose $v - k > x$ for all $x \in U$. Then $v - k$ is an upper bound for U which is less than the least upper bound of U. But this cannot happen. It follows that there exists an $x \in U$ such that $v - k < x$. Then $[0, v - k] \subset [0, x]$. Now since $x \in U$, there is a finite collection C' of elements of C* such that $[0, v - k] \subset [0, x] \subset C'$. Then $[0, v]$ is covered by the finite collection $C' \cup C_v$. Therefore $v \in U$.

Now since $v \in U$, there exist elements $C_1, C_2, \ldots C_n$ in C* such that $C_1 \cup C_2 \cup \ldots \cup C_n$ is a finite subcover for $[0, v]$. Then $C_1 \cup C_2 \cup \ldots \cup C_n \cup V$ is a finite subcover for $[0, v + \frac{\lambda}{2}] \subset [0, 1]$. So $v + \frac{\lambda}{2} \in U$. But this contradicts the definition of v as sup U.

Therefore it cannot happen that $v < 1$. Hence $v \geq 1$. Then $1 \in U$, so $[0, 1]$ can be covered by a finite number of elements of C*. Thus $([0, 1], T_{[0, 1]})$ is compact, so the closed unit interval is compact as a subset of the reals.

<u>Theorem 4</u>: The open unit interval is not compact.

Proof: Let (X, T) denote the reals with the usual topology, and consider $((0, 1), T_{(0, 1)})$.

Let $C^* = \left\{(\frac{1}{n+1}, \frac{1}{n-1}) \mid n = 2, 3, 4, \ldots\right\}$.

Then C* is an open cover for (0, 1). But C* has no finite subcover since if any interval $(\frac{1}{n+1}, \frac{1}{n-1})$ is removed, the point $\frac{1}{n}$ is left uncovered. Thus $((0, 1), T_{(0, 1)})$ is not compact, so the open unit interval is not compact as a subset of the reals.

<u>Theorem 5</u>: (a) $[0, 1)$ is not compact.

(b) $(0, 1]$ is not compact.

Proof: Let (X, T) denote the reals with the usual topology.

(a) Consider $([0, 1), T_{[0, 1)})$. The interval $(-\frac{1}{4}, \frac{1}{4})$ is open in X, so $(-\frac{1}{4}, \frac{1}{4}) \cap [0, 1) = [0, \frac{1}{4})$ is open in $[0, 1)$. Then $[0, \frac{1}{4}) \cup \left\{(\frac{1}{n+1}, \frac{1}{n-1}) \mid n = 2,\right.$

$3, 4, \ldots \}$ is an open cover for $[0, 1)$. But this open cover has no finite subcover since if $[0, \frac{1}{4})$ is removed, the point 0 is left uncovered; and if any interval $(\frac{1}{n+1}, \frac{1}{n-1})$ is removed, the point $\frac{1}{n}$ is left uncovered.

Therefore, $([0, 1), T_{[0, 1)})$ is not compact, so $[0, 1)$ is not compact as a subset of the reals.

(b) Consider $((0, 1], T_{(0, 1]})$. The interval $(\frac{1}{3}, \frac{4}{3})$ is open in the reals, so $(\frac{1}{3}, \frac{4}{3}) \cap (0, 1] = (\frac{1}{3}, 1]$ is open in $(0, 1]$. Then $(\frac{1}{3}, 1] \cup \left\{ (\frac{1}{n+1}, \frac{1}{n-1}) \mid n = 3, 4, 5, \ldots \right\}$ is an open cover for $(0, 1]$. But this open cover has no finite subcover since if the interval $(\frac{1}{3}, 1]$ is removed, the point 1 is left uncovered; and if any interval $(\frac{1}{n+1}, \frac{1}{n-1})$ is removed, the point $\frac{1}{n}$ is left uncovered.

Therefore, $((0, 1], T_{(0, 1]})$ is not compact, so $(0, 1]$ is not compact as a subset of the reals.

<u>Example 1</u>: Let (X, T) denote the reals with the usual topology.

(a) $([0, 1], T_{[0, 1]})$ is not homeomorphic to $([0, 1), T_{[0, 1)})$.

(b) $([0, 1], T_{[0, 1]})$ is not homeomorphic to $((0, 1], T_{(0, 1]})$.

Proof: (a) The interval $[0, 1]$ is compact, while the interval $[0, 1)$ is not. Therefore by Theorem 2, the spaces induced by these respective subsets of the reals are not homeomorphic.

(b) As before, $[0, 1]$ is compact, while $(0, 1]$ is not. Hence the two spaces induced by these sets are not homeomorphic.

Theorem 6: The reals with the usual topology are not compact.

Proof: Let $V = \left\{(n - 1, n + 1) \mid n \text{ is an integer}\right\}$. Then V is an open cover for the reals. But V has no finite subcover since if any interval $(n - 1, n + 1)$ is removed, the point n is left uncovered.

Therefore the reals with the usual topology are not compact.

Example 2: $[0, 1]$ and $(0, 1)$, each with the usual topology inherited from the reals, are not homeomorphic.

Proof: $[0, 1]$ is compact, while $(0, 1)$ is not. Therefore, by Theorem 2, the subspaces induced by these sets are not homeomorphic.

Example 3: Let (X, T) denote the reals with the usual topology. Then $(X, T) \backsim ((0, 1), T_{(0, 1)})$.

Proof: Define a mapping $f\colon (X, T) \rightarrow (0, 1)$ by

$$f(x) = \begin{cases} \frac{1}{2} + \frac{x}{2x + 1} & \text{if } x \geq 0 \\ \frac{1}{2} - \frac{x}{2x - 1} & \text{if } x < 0 \end{cases}$$

We need to show first that f does indeed map the reals into

(0, 1). We begin by showing that f', the derivative of f, does exist. Let $a \in X$. If $a > 0$, then

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\frac{1}{2} + \frac{x}{2x + 1} - \frac{1}{2} - \frac{a}{2a + 1}}{x - a}$$

$$= \lim_{x \to a} \frac{1}{(2x + 1)(2a + 1)}$$

$$= \frac{1}{(2a + 1)^2}$$

which is a real number since $a > 0$. If $a < 0$, then

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\frac{1}{2} - \frac{x}{2x - 1} - \frac{1}{2} + \frac{a}{2a - 1}}{x - a}$$

$$= \lim_{x \to a} \frac{1}{(2x - 1)(2a - 1)}$$

$$= \frac{1}{(2a - 1)^2}$$

which is a real number since $a < 0$. In the case where $a = 0$, we examine the limits as x approaches 0 from above and below.

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{\frac{1}{2} + \frac{x}{2x + 1} - \frac{1}{2}}{x}$$

$$= \lim_{x \to 0^+} \frac{1}{2x + 1}$$

$$= 1$$

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{\frac{1}{2} - \frac{x}{2x - 1} - \frac{1}{2}}{x}$$

$$= \lim_{x \to 0^-} - \frac{1}{2x - 1}$$

$$= 1$$

Then

$$\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^-} f'(x)$$

so the limit exists and thus the function is differentiable at 0.

Since in each case the derivative exists and is positive, by [2, Theorem 6.3, p. 129], f is an increasing function. Also, since we have shown that f is differentiable, we can conclude that f is continuous.

Now we must show that f is bounded below and above by 0 and 1, respectively. Let $x \in X$. If $x \geq 0$, then

$$2x < 2x + 1$$

$$\frac{x}{2x + 1} < \frac{1}{2}$$

$$\frac{1}{2} + \frac{x}{2x + 1} < 1$$

If $x < 0$, then

$$2x > 2x - 1$$

$$\frac{x}{2x - 1} < \frac{1}{2}$$

since $2x - 1$ is negative, and thus

$$\frac{1}{2} - \frac{x}{2x - 1} > \frac{1}{2} - \frac{1}{2} = 0$$

Then since f is increasing, f maps into (0, 1).

To show f is one-to-one, we consider two cases:

(1) Let $f(x_1)$ and $f(x_2) \epsilon (0, \frac{1}{2})$ such that

$$f(x_1) = f(x_2)$$

Then

$$\frac{1}{2} - \frac{x_1}{2x_1 - 1} = \frac{1}{2} - \frac{x_2}{2x_2 - 1}$$

$$-2x_1x_2 + x_1 = -2x_1x_2 + x_2$$

$$x_1 = x_2$$

(2) Let $f(x_1)$ and $f(x_2) \epsilon [\frac{1}{2}, 1)$ such that

$$f(x_1) = f(x_2)$$

Then

$$\frac{1}{2} + \frac{x_1}{2x_1 + 1} = \frac{1}{2} + \frac{x_2}{2x_2 + 1}$$

$$2x_1x_2 + x_1 = 2x_1x_2 + x_2$$

$$x_1 = x_2$$

Therefore f is one-to-one.

Next we must show that f is onto. Let $y \epsilon (0, 1)$. We must show that there exists an $x \epsilon X$ such that $f(x) = y$. Again there are two cases to consider:

(1) If $y \epsilon (0, \frac{1}{2})$, choose $x = \frac{y - \frac{1}{2}}{2y}$. Now $0 < y < \frac{1}{2}$, so $y - \frac{1}{2} < 0$, and $2y > 0$. Then $x = \frac{y - \frac{1}{2}}{2y} < 0$, and

$$f(x) = f\left(\frac{y - \frac{1}{2}}{2y} \right) = \frac{\frac{y - \frac{1}{2}}{2y}}{\frac{2y - 1}{2y} - 1}$$

$$= \frac{1}{2} - \left(\frac{y - \frac{1}{2}}{2y} \right)\left(\frac{2y}{2y - 1 - 2y} \right)$$

$$= \frac{1}{2} - \left(\frac{y - \frac{1}{2}}{-1} \right)$$

$$= y$$

(2) If $y \in [\frac{1}{2}, 1)$, choose $x = \dfrac{y - \frac{1}{2}}{2 - 2y}$. Now $\frac{1}{2} \leq y < 1$, so $2 - 2y > 0$ and $y - \frac{1}{2} \geq 0$. Thus $\dfrac{y - \frac{1}{2}}{2 - 2y} \geq 0$, and

$$f(x) = f\left(\frac{y - \frac{1}{2}}{2 - 2y} \right) = \frac{\frac{y - \frac{1}{2}}{2 - 2y}}{\frac{2y - 1}{2 - 2y} + 1}$$

$$= \frac{1}{2} + \left(\frac{y - \frac{1}{2}}{2 - 2y} \right)\left(\frac{2 - 2y}{2y-1+2-2y} \right)$$

$$= \frac{1}{2} + y - \frac{1}{2}$$

$$= y$$

Thus in either case we can find some $x \in X$ such that $f(x) = y$, and it follows that f is onto.

We next examine f^{-1}: $(0, 1) \to (X, T)$, defined by

$$f^{-1}(y) = \begin{cases} \dfrac{y - \frac{1}{2}}{2y} & \text{if } y \in (0, \frac{1}{2}) \\[2ex] \dfrac{y - \frac{1}{2}}{2 - 2y} & \text{if } y \in [\frac{1}{2}, 1) \end{cases}$$

First we show that f^{-1} is differentiable. Let $a \in (0, \frac{1}{2})$. Then

$$(f^{-1})'(a) = \lim_{y \to a} \frac{f^{-1}(y) - f^{-1}(a)}{y - a}$$

$$= \lim_{y \to a} \frac{\frac{y - \frac{1}{2}}{2y} - \frac{a - \frac{1}{2}}{2a}}{y - a}$$

$$= \lim_{y \to a} \frac{\frac{(y - \frac{1}{2})a - (a - \frac{1}{2})y}{2ay}}{y - a}$$

$$= \lim_{y \to a} \frac{1}{4ay}$$

$$= \frac{1}{4a^2}$$

which is a positive real number.

If $a \in (\frac{1}{2}, 1)$, then

$$(f^{-1})'(a) = \lim_{y \to a} \frac{\frac{y - \frac{1}{2}}{2 - 2y} - \frac{a - \frac{1}{2}}{2 - 2a}}{y - a}$$

$$= \lim_{y \to a} \left(\frac{\frac{1}{2}(y - a)}{(1 - y)(1 - a)}\right)\left(\frac{1}{y - a}\right)$$

$$= \lim_{y \to a} \frac{1}{2(1 - y)(1 - a)}$$

$$= \frac{1}{2(1 - a)^2}$$

which is a positive real number since $a \neq 1$.

For the case where $a = \frac{1}{2}$, we look at the limits as y approaches $\frac{1}{2}$ from above and below.

$$\lim_{y \to \frac{1}{2}^+} \frac{f^{-1}(y) - f^{-1}(\frac{1}{2})}{y - \frac{1}{2}} = \lim_{y \to \frac{1}{2}^+} \frac{\frac{y - \frac{1}{2}}{2 - 2y} - 0}{y - \frac{1}{2}}$$

$$= \lim_{y \to \frac{1}{2}^+} \frac{1}{2 - 2y}$$

$$= 1$$

$$\lim_{y \to \frac{1}{2}^-} \frac{f^{-1}(y) - f^{-1}(\frac{1}{2})}{y - \frac{1}{2}} = \lim_{y \to \frac{1}{2}^-} \frac{\frac{y - \frac{1}{2}}{2y} - 0}{y - \frac{1}{2}}$$

$$= \lim_{y \to \frac{1}{2}^-} \frac{1}{2y}$$

$$= 1$$

Then

$$\lim_{y \to \frac{1}{2}^+} \frac{f^{-1}(y) - f^{-1}(\frac{1}{2})}{y - \frac{1}{2}} = \lim_{y \to \frac{1}{2}^-} \frac{f^{-1}(y) - f^{-1}(\frac{1}{2})}{y - \frac{1}{2}}$$

and therefore f^{-1} is differentiable at $\frac{1}{2}$.

Now since f^{-1} has been shown to be differentiable, we may conclude that f^{-1} is continuous.

We have now shown that $f: (X, T) \to (0, 1)$ is continuous, one-to-one, and onto, and that f^{-1} is continuous. Hence f is a homeomorphism, and $(X, T) \backsim ((0, 1), T_{(0, 1)})$.

<u>Example 4</u>: Let (X, T) denote the reals with the usual topology. Then (X, T) is not homeomorphic to $([0, 1], T_{[0, 1]})$.

Proof: Suppose $(X, T) \backsim ([0, 1], T_{[0, 1]})$. Then since $(X, T) \backsim ((0, 1), T_{(0, 1)})$ by Example 3, and homeomorphism is an equivalence relation, we have $([0, 1], T_{[0, 1]}) \backsim ((0, 1), T_{(0, 1)})$. But this contradicts Example 2. Hence (X, T) cannot be homeomorphic to $([0, 1], T_{[0, 1]})$.

<u>Theorem 7</u>: Let (X, T) denote the reals with the usual topology. For every $a, b \in X$ such that $a < b$, we have

(1) $(a, b) \backsim (0, 1)$

(2) $[a, b] \backsim [0, 1]$

(3) $(a, b] \backsim (0, 1]$

(4) $[a, b) \backsim [0, 1)$

where each interval has the usual topology inherited from the reals.

Proof: (1) Define $f: (a, b) \to (0, 1)$ by

$$f(x) = \frac{x - a}{b - a} \text{ for each } x \in (a, b).$$

We first observe that f does map into $(0, 1)$. Let $x \in (a, b)$. Then $a < x < b$, so $0 < x - a < b - a$, and thus $0 < \frac{x - a}{b - a} < 1$ since $b - a > 0$. Then for each $x \in (a, b)$,

$\frac{x - a}{b - a} \in (0, 1)$, which implies that f maps into (0, 1). Clearly, f is well-defined since any value for x can yield exactly one value for $\frac{x - a}{b - a} = f(x)$.

We now show that f is one-to-one. Let $f(x_1)$ and $f(x_2) \in (0, 1)$ such that $f(x_1) = f(x_2)$, that is

$$\frac{x_1 - a}{b - a} = \frac{x_2 - a}{b - a}$$

$$x_1 - a = x_2 - a$$

$$x_1 = x_2$$

Therefore f is one-to-one.

We now prove that f is onto. Let $y \in (0, 1)$. We must show that there exists $x \in (a, b)$ such that $f(x) = y$.

Consider $x = a + y(b - a)$. We first show that this x is contained in (a, b). Since $y > 0$ and $b - a > 0$, it follows that $a + y(b - a) > a$. Also, since $y < 1$, then $a + y(b - a) < a + (b - a) = b$. Then $x = a + y(b - a) \in (a, b)$.

We now show that $f(x) = y$.

$$f(x) = \frac{x - a}{b - a} = \frac{a + y(b - a) - a}{b - a}$$

$$= \frac{y(b - a)}{b - a}$$

$$= y$$

Therefore f is onto.

We need to show now that f is continuous. Let $\epsilon > 0$ and $x \in (a, b)$. Let $\lambda = \epsilon(b - a)$ and $y \in (a, b)$. Then

$|x - y| < \lambda$ implies that

$$|f(x) - f(y)| = \left| \frac{x - a}{b - a} - \frac{y - a}{b - a} \right|$$
$$= \left| \frac{x - y}{b - a} \right|$$
$$= |x - y| \frac{1}{b - a} \quad \text{(since } b\text{-}a > 0)$$
$$< \lambda\left(\frac{1}{b - a} \right)$$
$$= \epsilon(b - a)\left(\frac{1}{b - a}\right)$$
$$= \epsilon$$

Thus f is continuous.

We now must show that f^{-1} is continuous. The function f^{-1} is defined by

$$f^{-1}(y) = a + y(b - a) \quad \text{for each } y \in (0, 1)$$

Let $\epsilon > 0$, $y \in (0, 1)$, $\lambda = \frac{\epsilon}{b - a}$, and $y_1 \in (0, 1)$. Then $|y - y_1| < \lambda$ implies that

$$|f(y) - f(y_1)| = |a + y(b - a) - a - y_1(b - a)|$$
$$= |b - a|\ |y - y_1|$$
$$< |b - a|(\lambda)$$
$$= (b - a)\left(\frac{\epsilon}{b - a}\right) \quad \text{since } b\text{-}a > 0$$
$$= \epsilon$$

Therefore f^{-1} is continuous, and it follows that f is a homeomorphism, that is, $(0, 1) \backsim (a, b)$.

It follows readily that (2), (3), and (4) are true, using the same mapping f extended to include endpoints of (a, b) and (0, 1) where necessary.

<u>Corollary 1</u>: Let (X, T) denote the reals with the usual topology.

(a) (X, T) is homeomorphic to the subspace induced by any open interval (a, b), where $a, b \in X$.

(b) (X, T) is not homeomorphic to any subspace induced by a closed interval $[a, b]$, where $a, b \in X$.

Proof: Let (X, T) denote the reals with the usual topology, and let $a, b \in X$ such that $a \angle b$.

(a) $(X, T) \backsim ((0, 1), T_{(0, 1)})$ by Example 3, and by Theorem 7 $((0, 1), T_{(0, 1)}) \backsim ((a, b), T_{(a, b)})$. Therefore, since $\backsim$ is an equivalence relation, we have $(X, T) \backsim ((a, b), T_{(a, b)})$ by the transitive property.

(b) Suppose that $(X, T) \backsim ([a, b], T_{[a, b]})$. By Theorem 7 we know $[a, b] \backsim [0, 1]$, and we showed in Theorem 3 that $[0, 1]$ is compact, so we may conclude that $[a, b]$ is compact also. Then X must be compact. But this contradicts Theorem 6. Therefore (X, T) cannot be homeomorphic to any subspace induced by a closed interval $[a, b]$.

CHAPTER III

CONNECTEDNESS

Definition 5: A topological space (X, T) is said to be connected provided X is not the union of two non-empty disjoint open subsets of X.

Definition 6: Let (X, T) be a topological space. A subset A of X is connected provided (A, T_A) is connected.

Theorem 8: The continuous image of a connected set is connected.

Proof: Let (X, S) and (Y, T) be topological spaces and f: $(X, S) \rightarrow (Y, T)$ be a continuous function. Let X be connected, and suppose f(X) is not connected. Then $f(X) = U \cup V$, where U and $V \in T$, neither U nor V is the empty set, and $U \cap V = \phi$.

Since f is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are open in X; furthermore, $f^{-1}(U) \cup f^{-1}(V) = X$ since $U \cup V = f(X)$. Clearly, $f^{-1}(U) \cap f^{-1}(V) = \phi$, for if $x \in f^{-1}(U) \cap f^{-1}(V)$, then it would follow that $f(x) \in U \cap V = \phi$, which cannot happen. Thus we have shown that X is the union of two disjoint non-empty open (in X) sets, which implies that X is not connected. But this contradicts the hypothesis of the theorem. Hence f(X) is connected.

<u>Corollary 2</u>: Connectedness is a topological property.

Proof: Let $f\colon (X, S) \to (Y, T)$ be a homeomorphism. Then $f(X) = Y$, $f^{-1}(Y) = X$, and both f and f^{-1} are continuous. It follows by Theorem 8 that if (X, S) is connected, then (Y, T) is connected; and if (Y, T) is connected, then (X, S) is connected. Thus connectedness is a topological property.

<u>Theorem 9</u>: The reals with the usual topology are connected.

Proof: Let (X, T) denote the reals with the usual topology. Suppose that X is not connected. Then $X = A \cup B$ where A and B are disjoint non-empty open (in X) sets. Let $a \in A$ and $b \in B$. Now $A \cap B = \phi$, so $a \neq b$; then either $a < b$, or $b < a$. Without loss of generality, we may assume $a < b$.

Let $C = A \cap [a, b]$, and let $c = \text{lub } C$. Obviously, for any element $x \in C$, it must be true that $a \leq x \leq b$. Then b is an upper bound for C, so that $\text{lub } C = c \leq b$.

Now c is a real number, so either $c \in A$ or $c \in B$. First suppose $c \in A$. Since A is open, there exists a real number $\lambda > 0$ such that $(c - \lambda, c + \lambda) \subset A$. Then

$$[c - \frac{\lambda}{2}, c + \frac{\lambda}{2}] \subset (c - \lambda, c + \lambda) \subset A$$

and thus $c - \frac{\lambda}{2} \in A$. We now show that $c + \frac{\lambda}{2} \in [a, b]$. We must prove that (1) $a \leq c + \frac{\lambda}{2}$, and that (2) $c + \frac{\lambda}{2} \leq b$.

(1) Since $a < c$, it follows clearly that $a < c + \frac{\lambda}{2}$.

(2) Suppose $c + \frac{\lambda}{2} > b$. Then $b \in [c, c + \frac{\lambda}{2}) \subset A$, that is, $b \in A$. But b cannot be an element of A since $b \in B$ and $A \cap B = \phi$. Hence $c + \frac{\lambda}{2} \leq b$.

Then $c + \frac{\lambda}{2}$ is an element of both A and $[a, b]$, so $c + \frac{\lambda}{2} \in A \cap [a, b] = C$. But then $c + \frac{\lambda}{2}$ is an element of C which is greater then the least upper bound of C, and therefore the assumption that $c \in A$ is false.

It follows that $c \in B$. Now B is open, so there is a real number $\epsilon > 0$ such that $(c - \epsilon, c + \epsilon) \subset B$. Then

$$[c - \frac{\epsilon}{2}, c + \frac{\epsilon}{2}] \subset (c - \epsilon, c + \epsilon) \subset B$$

so $c - \frac{\epsilon}{2}$ is in B. Then $c - \frac{\epsilon}{2}$ is not an element of A, and hence not an element of C. But $c - \frac{\epsilon}{2}$ is an element of B and thus is an upper bound for C which is less than the least upper bound for C. Since this cannot happen, the assumption that $c \in B$ is false.

Then c is a real number which is contained in neither A nor B, contradicting our assumption that $A \cup B = X$, and therefore implying that X is connected.

Theorem 10: The rationals with the usual subspace topology are not connected.

Proof: Let X denote the reals and (Q, T) the rationals with the usual induced topology. Let $x \in X$ and consider $\{x \mid x < \sqrt{2}\} \cap Q$ and $\{x \mid x > \sqrt{2}\} \cap Q$. Each of these two sets is open in Q and each is non-empty. Also,

$$(\{x \mid x > \sqrt{2}\} \cap Q) \cap (\{x \mid x < \sqrt{2}\} \cap Q)$$
$$= (\{x \mid x < \sqrt{2}\} \cap \{x \mid x > \sqrt{2}\}) \cap Q$$
$$= \phi \cap Q$$
$$= \phi$$

and

$$(\{x \mid x > \sqrt{2}\} \cap Q) \cup (\{x \mid x < \sqrt{2}\} \cap Q)$$
$$= (\{x \mid x > \sqrt{2}\} \cup \{x \mid x < \sqrt{2}\}) \cap Q$$
$$= (X - \{\sqrt{2}\}) \cap Q$$
$$= Q$$

Then Q is the union of two non-empty disjoint open sets, and thus Q is not connected.

Example 5: The reals with the usual topology are not homeomorphic to the rationals with the usual induced topology.

Proof: Let (X, T) denote the reals with the usual topology and let Q denote the rationals. Suppose that $(X, T) \backsim (Q, T_Q)$. Then since (X, T) is connected and connectedness is a topological property, we may conclude that (Q, T_Q) is connected. But this contradicts Theorem 10; therefore our original assumption was false. Hence the reals and the rationals are not homeomorphic.

Theorem 11: Any interval is connected.

Proof: Let X be an interval, and suppose that X is not connected. Then $X = A \cup B$, where A and B are open in X, neither A nor B is the empty set, and $A \cap B = \phi$.

Let $a \in A$, and $b \in B$. We know $a \neq b$ since A and B are disjoint, so either $a \angle b$ or $b \angle a$. Without loss of generality we may assume that $a \angle b$.

Let $V = \left\{ x \in [a, b] \mid [a, x) \subset A \right\}$, and let $v = \text{lub } V$. Now $v = a$ would imply that v is the largest element of A, and A has no largest element since A is open. Then $a \angle v \underline{\angle} b$. We know $v \in \overline{A_X}$, the closure of A in X, and A is the complement in X of B. But B is open, so A is closed. Thus $A = \overline{A_X}$, and $v \in A$.

Now A is also open, so there exists a real number $\lambda > 0$ such that $(v - \lambda, v + \lambda) \subset A$. Then $[a, v + \lambda) \subset A$, and hence $v + \lambda \in V$. But then $v + \lambda$ is an element of V which is greater then the least upper bound for V, which is a contradiction. Then our assumption that X is not connected was incorrect, and it follows that any interval is connected.

<u>Example 6</u>: Removing an interior point from one of two homeomorphic intervals, as illustrated in Figure 1 below, destroys the homeomorphism.

Figure 1

Proof: We may assume without loss of generality that the intervals under consideration are closed intervals; the proof given is valid for both open and half-open intervals.

Since $[a, b]$ is an interval, it is connected by Theorem 11. But $\{[c, d] - \{p\}\}$ is not connected since

$$\{[c, d] - \{p\}\} = [c, p) \cup (p, d]$$

where both $[c, p)$ and $(p, d]$ are open in the subspace $[c, d]$ of the reals; each of the sets $[c, p)$ and $(p, d]$ is non-empty, and $[c, p) \cap (p, d] = \phi$.

Then since $[a, b]$ is connected and $\{[c, d] - \{p\}$ is not, it follows that the two spaces induced by these sets are not homeomorphic.

CHAPTER IV

POINTS OF SEPARATION

Definition 7: Let X be a connected set with topology T, and let $p \in X$. Then p is a point of separation of X if $X - \{p\}$ is not connected.

Theorem 12: The property of having a point of separation is a topological property.

Proof: Let X and Y be connected sets with topologies S and T, respectively, and let $f\colon (X, S) \to (Y, T)$ be a homeomorphism. Let p be a point of separation for X. Then $X - \{p\}$ is not connected. Now $f(X - \{p\}) = f(X) - \{f(p)\} = Y - \{f(p)\}$. Suppose that $Y - \{f(p)\}$ is connected. Then since f^{-1} is continuous, $f^{-1}(Y - \{f(p)\})$ must be connected by Theorem 8. But $f^{-1}(Y - \{f(p)\}) = X - \{p\}$, which is not connected. Therefore our supposition that $Y - \{f(p)\}$ is connected was false, so $f(p)$ is a point of separation of Y. Thus if X has a point of separation, then Y has a point of separation.

The proof that if Y has a point of separation, then X has a point of separation is entirely analogous. It follows that the property of having a point of separation is a topological property.

Definition 8: Let (X, T) be a topological space and let A and B be non-empty subsets of X. Then A and B are said to be mutually separated provided that $A \cap \overline{B_X} = \phi$ and $\overline{A_X} \cap B = \phi$.

Theorem 13: Let (X, T) be a topological space. Then X is connected if and only if X is not the union of two mutually separated sets.

Proof: Let (X, T) be a topological space such that X is connected. Suppose $X = A \cup B$, where A and B are mutually separated. Then $\overline{A_X} \cap B = \phi$ and $A \cap \overline{B_X} = \phi$. Since $A \subset \overline{A_X}$ and $B \subset \overline{B_X}$, it follows that $A \cap B = \phi$. By definition, neither A nor B is the empty set. We now show that A and B are open in X.

First suppose that A is not closed. Then there is an element $x \in \overline{A_X}$ such that x is not an element of A. Now since x is not an element of A, it must be true that $x \in B$; so $x \in \overline{A_X} \cap B$. But this cannot happen since $\overline{A_X} \cap B = \phi$. Hence, A is closed, and it follows that B is open since B is the complement in X of A.

Similarly, A is open. Then X is the union of two disjoint non-empty open sets, and hence is not connected. But this contradicts the hypothesis that X is connected, so it cannot be true that X is the union of two mutually separated sets.

Now we must show that if X is not the union of two mutually separated sets, then X is connected. Assume that X is not the union of two mutually separated sets, and suppose X is not connected. Then $X = A \cup B$, where A and B are open in X, neither A nor B is the empty set, and A and B are disjoint. Since A is open in X and B is the complement of A in X, B must be closed in X. Then $B = \overline{B_X}$. Similarly, $A = \overline{A_X}$, and since $A \cap B = \phi$, we can conclude that $A \cap \overline{B_X} = \phi$ and $\overline{A_X} \cap B = \phi$. Thus A and B are mutually separated. But this contradicts the hypothesis that X is not the union of two mutually separated sets. Then the supposition that X is not connected was false. Hence if X is not the union of two mutually separated sets, then X is connected.

<u>Lemma 1</u>: Let (X, T) be a topological space, and let A and B be two mutually separated sets in X. Suppose $C \subset A \cup B$ such that C is connected. Then either $C \subset A$ or $C \subset B$.

Proof: Suppose that both $C \cap A \neq \phi$ and $C \cap B \neq \phi$ are true. Since A and B are mutually separated, we know that $A \cap \overline{B_X} = \phi$, $\overline{A_X} \cap B = \phi$, and both A and B are non-empty. Now

$$\begin{aligned} \overline{(C \cap A)_X} \cap (C \cap B) &\subset \overline{C_X} \cap \overline{A_X} \cap C \cap B \\ &= \overline{C_X} \cap C \cap (\overline{A_X} \cap B) \\ &= C \cap \phi = \phi \end{aligned}$$

Then $\overline{(C \cap A)}_X \cap (C \cap B) = \phi$. Similarly, $(C \cap A) \cap \overline{(C \cap B)}_X \subset \phi$, so $(C \cap A) \cap \overline{(C \cap B)}_X = \phi$, so we have shown that $C \cap A$ and $C \cap B$ are mutually separated in X. Now $\overline{(C \cap A)}_C \subset \overline{(C \cap A)}_X$ and $\overline{(C \cap B)}_C \subset \overline{(C \cap B)}_X$, so $C \cap A$ and $C \cap B$ are mutually separated in C. But $C \subset (A \cup B)$, so

$$\begin{aligned}(C \cap A) \cup (C \cap B) &= C \cap (A \cup B)\\ &= C\end{aligned}$$

so C is the union of two mutually separated sets in C. But C is connected, so we have contradicted Theorem 13. Therefore it must be true that either $C \cap A = \phi$ or $C \cap B = \phi$, that is, either $C \subset B$ or $C \subset A$.

Lemma 2: The union of two connected sets having at least one point in common is connected.

Proof: Let (Z, T) be a topological space and X and Y connected subsets of Z such that X and Y are not disjoint. Suppose that $X \cup Y$ is not connected. Then $X \cup Y = A \cup B$, where A and B are are mutually separated in Z.

Since X is connected, either $X \subset A$ or $X \subset B$, by Lemma 1. Without loss of generality, we may assume $X \subset A$. Similarly, Y is connected, so either $Y \subset A$ or $Y \subset B$. Suppose $Y \subset B$. Then since $A \cap B = \phi$, it follows that $A \cap Y = \phi$, so $X \cap Y = \phi$. But this is a contradiction to the hypothesis, which states that X and Y are not disjoint. Then Y is not a subset of B, so Y must be a subset of A. Hence $X \cup Y \subset A$, and thus $B = \phi$. But by definition, B cannot be the empty set, so we have reached a contradiction. Then our assumption that $X \cup Y = A \cup B$, where A and B are mutually separated, was incorrect. Hence $X \cup Y$ is connected.

Theorem 14: The plane has no point of separation.

Proof: Let E represent the plane. We must show that for any $p \in E$, $E - \{p\}$ is connected.

Suppose $E - \{p\}$ is not connected. Then $E - \{p\} = A \cup B$, where A and B are mutually separated. Let $x \in A$. Clearly, $x \neq p$ since $A \subset E - \{p\}$. Let $y \in E - \{p\}$, and let k denote the line segment from x to y. Choose any point z not on k; let m denote the line segment from z to x and n the line segment from z to y, as illustrated in Figure 2 below.

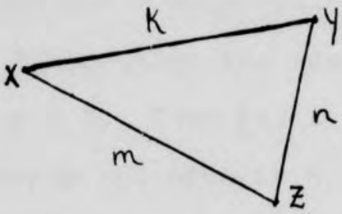

Figure 2

It is geometrically clear that p can belong to at most one of the sets k and $m \cup n$. If $p \in k$, then $m \cup n$ is a connected set containing x and y since each of m and n is connected (Lemma 2). From Lemma 1 and the fact that $x \in A$, it follows that $m \cup n \subset A$. Since $y \in m \cup n$, we have shown that for any element $y \in E - \{p\}$, $y \in A$. Hence $E - \{p\} \subset A$, which implies that $B = \phi$. But by definition of mutually separated, $B \neq \phi$.

If $p \in m \cup n$, then k is a connected set containing x and y, and as before we are led to the conclusion that $E - \{p\} \subset A$, implying that $B = \phi$, which cannot happen.

If p is contained in neither k nor $m \cup n$, then both sets are connected sets containing x and y, and we again conclude that $E - \{p\} \subset A$ and hence $B = \phi$.

In either case, the assumption that $E - \{p\}$ is not connected leads us to a contradiction. Therefore $E - \{p\}$ is connected, and thus the plane has no point of separation.

Example 7: The plane and the reals, each with its usual topology, are not homeomorphic.

Proof: Let X represent the reals and E the plane. Choose any point $p \in X$. Then $\{x \mid x \angle p\} \cap \{x \mid x \searrow p\} = \phi$, each set is non-empty and open in X, and $\{x \mid x \angle p\} \cup \{x \mid x \searrow p\} = X - \{p\}$. Thus $X - \{p\}$ is not connected, so p is a point of separation for the reals. We have just seen that the plane has no point of separation, however, so by Theorem 12 the reals and the plane cannot be homeomorphic.

Example 8: The sets A and B, as illustrated in Figure 3, each having the usual topology inherited from the plane, are not homeomorphic.

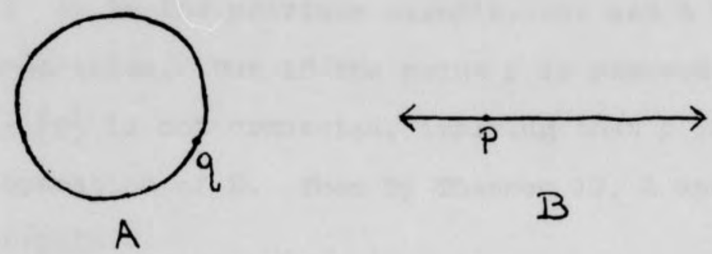

Figure 3

Proof: It is evident from the figure that the set A has no point of separation, since if any point q is removed, A - $\{q\}$ is still connected. But if any point p is removed from B, then B - $\{p\}$ is not connected, and hence p is a point of separation of B. Therefore, by Theorem 12, A and B are not homeomorphic.

Example 9: The sets A and B, as illustrated in Figure 4, each with the usual topology inherited from the plane, are not homeomorphic.

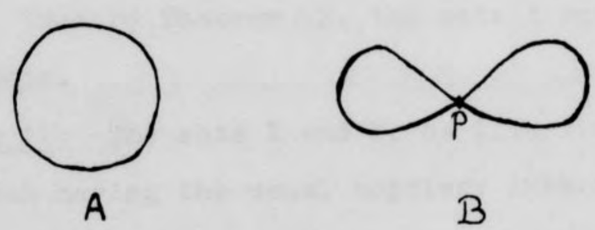

Figure 4

Proof: As in the previous example, the set A has no point of separation. But if the point p is removed from B, then B - $\{p\}$ is not connected, implying that p is a point of separation of B. Then by Theorem 12, A and B are not homeomorphic.

Example 10: The sets A and B, as illustrated in Figure 5, each having the usual topology inherited from the plane, are not homeomorphic.

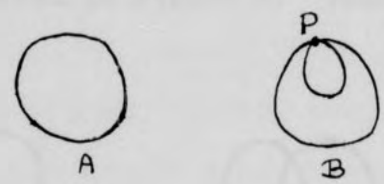

Figure 5

Proof: As before, the set A has no point of separation. However, it the point p is removed from B, then B - $\{p\}$ is not connected, and thus p is a point of separation for B. Then by Theorem 12, the sets A and B cannot be homeomorphic.

Example 11: The sets X and Y, as illustrated in Figure 6, each having the usual topology inherited from the plane, are not homeomorphic.

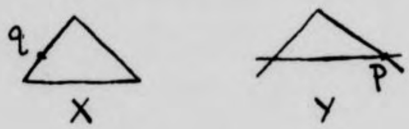

Figure 6

Proof: The set X has no point of separation, since if any point q is removed, then $X - \{q\}$ is still connected. But p is a point of separation for Y since $Y - \{p\}$ is clearly disconnected. It follows by Theorem 12 that X and Y are not homeomorphic.

<u>Example 12</u>: Let C be the set consisting of two circles which are tangent to each other, and let D be the set consisting of two circles which intersect at exactly two points, as illustrated in Figure 7. Then C and D are not homeomorphic.

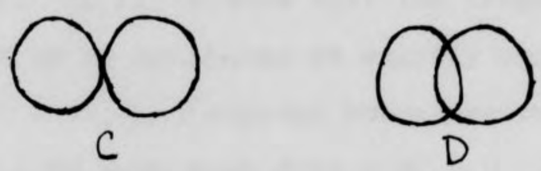

Figure 7

Proof: In set C, the point p at which the circles are tangent is clearly a point of separation of C. But in set D, we may remove a point of intersection or any other point of either circle and D remains a connected set. Therefore, by Theorem 12, C and D are not homeomorphic.

CHAPTER V
COMPONENTS

Definition 9: A component is a maximal connected set.

Theorem 15: Let f: $(X, S) \to (Y, T)$ be a homeomorphism and let $p \in X$. Let C^* denote the collection of components of $X - \{p\}$, and D^* the collection of components of $Y - \{f(p)\}$. Then there is a one-to-one correspondence between C^* and D^*.

Proof: We first show that the image under f of each element of C^* is contained in exactly one element of D^*. Consider $C \in C^*$, and suppose there are two distinct elements D_n and $D_m \in D^*$ such that $f(C) \subset D_n \cup D_m$. Now f(C) is connected since f is continuous, but $D_n \cup D_m$ is not connected since each of D_n and D_m is a maximal connected set. Then either $f(C) \subset D_n$ or $f(C) \subset D_m$. Thus f(C) is contained in exactly one element of D^*.

Now define a mapping λ such that λ associated each $C \in C^*$ to the element $D \in D^*$ such $f(C) \subset D$. We wish to show that λ is one-to-one. We have just seen that λ associates each element of C^* to exactly one element of D^*, so it remains to show that λ associates each element of D^* to exactly one element of C^*.

Suppose λ associates some element D_k of D^* to more than one element of C^*. Then since f is a homeomorphism, f is onto; so for each element d of D_k, $f^{-1}(d) \in X$ and thus in component C_d of $X - \{p\}$. (We know that $f^{-1}(d)$ is not p since $d \in Y - \{f(p)\}$.) Therefore, $f^{-1}(D_k) = C_i \cup C_j \cup C_k \cup \ldots$. Now $f^{-1}(D_k)$ is connected since f^{-1} is continuous, so $C_i \cup C_j \cup C_k \cup \ldots$ is connected. But this cannot happen since each element of C^* is maximal, and thus the union of any two elements cannot be connected. Therefore, our assumption that λ associates D_k to more than one element of C^* was false. Hence λ: $C^* \rightarrow D^*$ is one-to-one, so between the components of $X - \{p\}$ and $Y - \{f(p)\}$ there is a one-to-one corresponcence.

Example 13: The sets A and B, as illustrated in Figure 8, each with the usual topology inherited from the plane, are not homeomorphic.

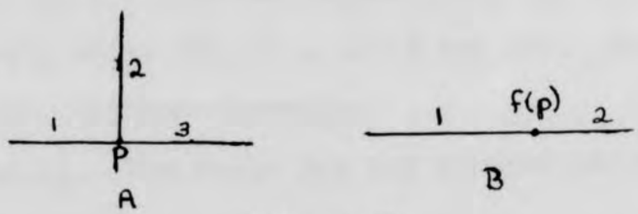

Figure 8

Proof: Suppose f: $A \rightarrow B$ is a homeomorphism, and consider the point $p \in A$. The set $A - \{p\}$ has three components, as numbered in the figure, while $B - \{f(p)\}$ has at most two.

(In the case where $f(p)$ is an endpoint of B, $B - \{f(p)\}$ has only one component.) But by Theorem 15, there must be a one-to-one correspondence between the components of $A - \{p\}$ and $B - \{f(p)\}$. Thus our assumption that f is a homeomorphism was false, and hence A is not homeomorphic to B.

<u>Example 14</u>: (a) $[0, 1)$ is not homeomorphic to $(0, 1)$

(b) $(0, 1]$ is not homeomorphic to $(0, 1)$

Proof: (a) Suppose $f: [0, 1) \to (0, 1)$ is a homeomorphism. Then $f(0) \in (0, 1)$. Now $[0, 1) - \{0\}$ has one component, namely $(0, 1)$. But $(0, 1) - \{f(0)\}$ has two: $(0, f(0))$ and $(f(0), 1)$. But by theorem 15, $[0, 1) - \{0\}$ and $(0, 1) - \{f(0)\}$ must have the same number of components. Thus the assumption that f is a homeomorphism leads to a contradiction, so $[0, 1)$ is not homeomorphic to $(0, 1)$.

(b) Suppose $h: (0, 1] \to (0, 1)$ is a homeomorphism. Then $h(1) \in (0, 1)$. But as in part (a), $(0, 1] - \{1\}$ has one component, while $(0, 1) - \{h(1)\}$ has two. Then $(0, 1]$ and $(0, 1)$ are not homeomorphic.

<u>Example 15</u>: The reals are not homeomorphic to the non-negative reals.

Proof: Suppose there is a homeomorphism $f: [0, \infty) \to (-\infty, \infty)$. Then consider the point 0. The set $[0, \infty) - \{0\}$ has one component, namely $(0, \infty)$, while $(-\infty, f(0))$ and $(f(0), \infty)$ are two components of $(-\infty, \infty)$. But by Theorem 15, the two sets must have the same number of components since f is a homeomorphism. Then the supposition that f is a homeomorphism is false, and the reals are not homeomorphic to the non-negative reals.

Example 16: A set whose points form two intersecting lines is not homeomorphic to a set whose points form two (Euclidean) parallel lines.

Proof: Let A denote the set whose points form two intersecting lines and B the set whose points form two parallel lines, as illustrated in Figure 9.

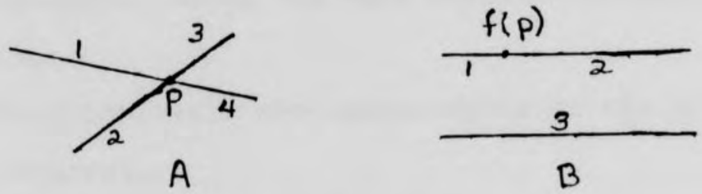

Figure 9

Suppose there is a homeomorphism $f: A \to B$. Let p denote the point of intersection of the two lines in A. Then $A - \{p\}$ has four components, as numbered, while $B - \{f(p)\}$ has three. But by Theorem 15, $A - \{p\}$ and $B - \{f(p)\}$ must have the same number of components since f is a homeomorphism. Thus the assumption that f is a homeomorphism is false, and A and B are not homeomorphic.

SUMMARY

In conclusion, it must be said that there remain other important properties which should be considered in a complete investigation of topological spaces, among these being local connectedness and local compactness. However, the four properties presented here are sufficient for making decisions about the topological equivalence of the spaces which are most frequently encountered in a study of topology. Among the most useful conclusions are the following:

that the reals are homeomorphic to the open unit interval,

that the reals are not homeomorphic to the closed unit interval nor to the rationals,

that any two open intervals are homeomorphic,

that any two closed intervals are homeomorphic,

that no two of the following are homeomorphic: an open interval, a closed interval, a half-open interval, and

that the reals are not homeomorphic to the plane.

Many additional conclusions may be obtained by combining these results and the others appearing in this paper.

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